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On the Onset of Breakup in Inviscid and Viscous Jets

by

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Abstract. This paper is concerned with the instability of inviscid and viscous jets utilizing the basic equations of the one-dimensional direct theory of a fluid jet based on the concept of a Cosserat (or a directed) curve. First, a system of differential equations is derived for small motions superposed on uniform flow of an inviscid straight circular jet which can twist along its axis. Periodic wave solutions are then obtained for this system of linear equations; and, with reference to a description of growth in the unstable mode, the comparison of the resulting dispersion relation is found to agree extremely well with the classical (three-dimensional) results of Rayleigh. Next, constitutive equations are obtained for a viscous elliptical jet and these are used to discuss both the symmetric and the anti-symmetric small disturbances in the shape of the free surface of a circular jet. Through a comparison with available three-dimensional numerical results, the solution obtained is shown to be an improvement over an existing approximate solution of the problem.

1. Introduction

The disintegration or breakup of a fluid jet due to surface tension, or so-called capillary instability, has long been a subject of interest in fluid mechanics. We do not include here a complete list of references on the subject, but mention those that have some bearing on the present paper. The classical study for an inviscid jet is due to Lord Rayleigh [1,2], who in turn cites the static investigations of Plateau and the experiments of Savart. Later Rayleigh [3] extended his earlier work to include viscosity but, due to analytical difficulty, explicitly considered only the case of negligible inertia. Weber [4] also examined the stability of a viscous jet and sought to remove the difficulty encountered in Rayleigh's work [3] by introducing approximations to the three-dimensional theory that take account of the 'thinness' of the jet. These approximations are *ad hoc* in nature and somewhat inconsistent, but they lead to results which agree reasonably well with Rayleigh's [1,2] in the special case of an inviscid jet.

In the references cited above, the problem of jet instability has been approached by considering small perturbations to a uniform cylindrical jet in the context of the linearized three-dimensional equations.* This procedure leads to a relatively simple dispersion relation in the case of an inviscid jet [1,2], but when the fluid is viscous the frequency of the wave motion is governed by an implicit transcendental equation and many results must be obtained numerically (see Chandrasekhar [6]) or by approximation [4]. Rather than consider another such approximation here we approach the subject using a one-dimensional theory of a directed (or a Cosserat) curve in the form given by Green, Naghdi and Wenner [7]. The relevance and applicability of this approach to problems involving fluid jets have been demonstrated in several papers by Green and

*These references consider the temporal instability of an infinite jet. This should be distinguished from the so-called spatial instability of a semi-infinite jet considered by Keller *et al.* [5].

Laws [8], by Green [9,10] and by Caulk and Naghdi [11]. Additional background on the theory of a Cosserat curve and its applications can be found in Green *et al*. [7,12].

With the use of the exact linearized three-dimensional equations, Rayleigh [1,2] derived the explicit result that the jet is unstable only in the axisymmetric mode of disturbance. Inasmuch as the present development does not begin with the three-dimensional equations, all modes of disturbance which occur in the present one-dimensional direct theory must be examined for stability. It is because of this that in section 2 of the paper we begin with a brief review of the basic equations for a straight jet of elliptical cross-section which can twist along its axis.† In section 3, we derive linearized field equations for an incompressible inviscid jet appropriate for small motions superposed on uniform flow of a circular jet. We show that the solution to these linearized equations can be decomposed into two modes, representing a symmetric and an anti-symmetric disturbance in the shape of the free surface. The anti-symmetric mode is stable for all wavelengths, while the symmetric mode is found to be unstable over a range of longer wavelengths. These conclusions are then compared with the corresponding exact three-dimensional analysis of Rayleigh [1,2].§ In terms of a description of growth in the unstable mode, the agreement in this case is remarkably good.

The constitutive equations for a linear viscous elliptical jet in the absence of twist are considered in section 4 and are utilized in section 5

†Consideration of a twisted elliptical jet will permit growth of a general disturbance which is not necessarily symmetric. Indeed, if *a priori* assumptions are made to restrict the motion of the jet, any conditions for instability (in the context of the present theory) are only sufficient.

§Here our results partly overlap with a recent study by Bogy [13], who deals only with axially symmetric disturbances of a nonrotating jet. In particular, Bogy considers the instability of an incompressible liquid viscous jet of circular section and in the main his work is concerned with spatial instability of a semi-infinite jet formulated as a boundary value problem.

to examine the stability of a cylindrical viscous jet. The results of this analysis are compared with those of Weber [4] and are shown to agree more closely with the three-dimensional numerical results of Chandrasekhar [6].

2. Basic equations.

We summarize in this section the main kinematics and differential equations characterizing the motion of a directed fluid jet in the form derived by Caulk and Naghdi [11]. The jet is straight, incompressible and homogeneous. Recall that this characterization of fluid jets is based on the concept of a Cosserat (or a directed) curve, hereafter designated as $\mathcal{R}$. Such a one-dimensional directed medium comprises a material line and a pair of directors attached to every point of the material line.

Let the particles of the material line of $\mathcal{R}$ be identified with the convected coordinate ξ; let c, the curve occupied by the material line of $\mathcal{R}$ in the present configuration at time t, be described by its position vector $\underset{\sim}{r}$ relative to a fixed origin; and let d_α $(\alpha = 1,2)$ stand for the pair of directors at $\underset{\sim}{r}$. Then, a motion of the directed curve $\mathcal{R}$ is specified by

$$\underset{\sim}{r} = \underset{\sim}{r}(\xi,t) \quad , \quad \underset{\sim}{d}_\alpha = \underset{\sim}{d}_\alpha(\xi,t) \quad . \tag{1}$$

The velocity and the director velocities are defined by

$$\underset{\sim}{v} = \dot{\underset{\sim}{r}} \quad , \quad \underset{\sim}{w}_\alpha = \dot{\underset{\sim}{d}}_\alpha \quad , \tag{2}$$

where a superposed dot designates the material time derivative holding ξ fixed.

For the purpose of displaying the details of the kinematics of a straight jet, including the rotation of the directors in a plane normal to the jet axis, we introduce a fixed system of rectangular Cartesian coordinates (x,y,z) with the z-axis parallel to the jet. Further, let the unit base vectors of the rectangular Cartesian axes be denoted by $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ and introduce, for later convenience, the additional base vectors

$$\underset{\sim}{e}_1 = \underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta \quad , \quad \underset{\sim}{e}_2 = -\underset{\sim}{i}\sin\theta + \underset{\sim}{j}\cos\theta \quad , \quad \underset{\sim}{e}_3 = \underset{\sim}{k} \quad , \tag{3}$$

where θ is a smooth function of z and t. We assume that the directors are so

restricted that they describe an elliptical cross-section of smoothly varying orientation along the length of the jet and that at each $z = \text{const.}$, the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ lie along the major and minor axes of the ellipse, respectively. Then, the angle θ, called the <u>sectional orientation</u>, specifies the orientation of the cross-section as a function of position. With this background, henceforth we restrict motions of the directed curve $\mathcal{R}$ such that in the present configuration at time t,

$$\mathbf{r} = z(\xi,t)\mathbf{e}_3 \quad , \quad \mathbf{d}_1 = \phi_1\mathbf{e}_1 \quad , \quad \mathbf{d}_2 = \phi_2\mathbf{e}_2 \tag{4}$$

where ϕ_1 and ϕ_2 measure the semiaxes of the elliptical cross-section. The velocity, the acceleration, the director velocities and the director accelerations assume the form†

$$\mathbf{v} = v\mathbf{e}_3 \quad , \quad v = \dot{z} \quad , \quad \dot{\mathbf{v}} = \dot{v}\mathbf{e}_3 \quad , \tag{5}$$

$$\mathbf{w}_1 = \phi_1(\zeta_1\mathbf{e}_1 + \omega_1\mathbf{e}_2) \quad , \quad \mathbf{w}_2 = \phi_2(\zeta_2\mathbf{e}_2 - \omega_2\mathbf{e}_1) \tag{6}$$

and

$$\begin{aligned} \dot{\mathbf{w}}_1 &= (\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2)\phi_1\mathbf{e}_1 + (2\zeta_1\omega_1 + \dot{\omega}_1)\phi_1\mathbf{e}_2 \quad , \\ \dot{\mathbf{w}}_2 &= (\dot{\zeta}_1 + \zeta_2^2 - \omega_2^2)\phi_2\mathbf{e}_2 - (2\zeta_2\omega_2 + \dot{\omega}_2)\phi_2\mathbf{e}_1 \quad , \end{aligned} \tag{7}$$

where

$$\dot{\phi}_1 = \phi_1\zeta_1 \quad , \quad \dot{\phi}_2 = \phi_2\zeta_2 \tag{8}$$

and ω_1 and ω_2 represent the rotational components of the director velocities in the plane normal to the jet axis.

The condition expressing the incompressibility of the fluid medium is given by

$$v_z + \zeta_1 + \zeta_2 = 0 \tag{9}$$

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†It should be noted that (6) and (7) represent values for $\mathbf{w}_\alpha$ and $\dot{\mathbf{w}}_\alpha$ in the present configuration and are not obtained by direct differentiation of the corresponding present values $(4)_{2,3}$ for $\mathbf{d}_\alpha$; see, in this connection, Caulk and Naghdi [11].

and the differential equations of motion for the elliptical jet described above are (see Caulk and Naghdi [11]):

$$\frac{\partial \hat{\underset{\sim}{n}}}{\partial z} = \{p_z + \phi_2\phi_{1z}h(\phi_1,\phi_2,\theta_z) + \phi_1\phi_{2z}h(\phi_2,\phi_1,\theta_z) + \theta_z(\phi_2^2-\phi_1^2)m(\phi_1,\phi_2,\theta_z)$$

$$+ \pi\rho^*\phi_1\phi_2\dot{v}\}\underset{\sim}{e}_3 \quad , \tag{10}$$

$$\left(\frac{\partial \hat{\underset{\sim}{p}}^1}{\partial z} - \hat{\underset{\sim}{\pi}}^1/z'\right)\phi_1 = \{-p - \phi_1\phi_2 h(\phi_1,\phi_2,\theta_z) + \tfrac{1}{4}\pi\rho^*\phi_1^3\phi_2(\dot{\zeta}_1 + \zeta_1^2 - \omega_1^2)\}\underset{\sim}{e}_1$$

$$+ \{\phi_1^2 m(\phi_1,\phi_2,\theta_z) + \tfrac{1}{4}\pi\rho^*\phi_1^3\phi_2(\dot{\omega}_1 + 2\zeta_1\omega_1)\}\underset{\sim}{e}_2 \quad , \tag{11}$$

$$\left(\frac{\partial \hat{\underset{\sim}{p}}^2}{\partial z} - \hat{\underset{\sim}{\pi}}^2/z'\right)\phi_2 = \{-p - \phi_1\phi_2 h(\phi_2,\phi_1,\theta_z) + \tfrac{1}{4}\pi\rho^*\phi_2^3\phi_1(\dot{\zeta}_2 + \zeta_2^2 - \omega_2^2)\}\underset{\sim}{e}_2$$

$$- \{\phi_2^2 m(\phi_2,\phi_1,\theta_z) + \tfrac{1}{4}\pi\rho^*\phi_2^3\phi_1(\dot{\omega}_2 + 2\zeta_2\omega_2)\}\underset{\sim}{e}_1 \quad , \tag{12}$$

$$z'\underset{\sim}{e}_3 \times \hat{\underset{\sim}{n}} + \underset{\sim}{d}_\alpha \times \hat{\underset{\sim}{\pi}}^\alpha + \frac{\partial \underset{\sim}{d}_\alpha}{\partial \xi} \times \hat{\underset{\sim}{p}}^\alpha = 0 \tag{13}$$

where $z' = \partial z/\partial\xi$ and a subscripted z denotes $\partial/\partial z$. The kinetic quantities $\hat{\underset{\sim}{n}}$, $\hat{\underset{\sim}{\pi}}^\alpha$ and $\hat{\underset{\sim}{p}}^\alpha$ are specified by constitutive equations, p is related to an average pressure over the cross-section of the jet and ρ^* is the three-dimensional density of the fluid. In addition, h and m arise due to the constant surface tension T and are given by

$$h(\phi_1,\phi_2,\theta_z) = \int_0^{2\pi} q\cos^2\chi\, d\chi \quad , \tag{14}$$

$$m(\phi_1,\phi_2,\theta_z) = -\tfrac{1}{2}\int_0^{2\pi} q\sin 2\chi\, d\chi \quad , \tag{15}$$

where

$$q = T\{[\theta_z \sin\chi\cos\chi\,(\phi_2^2-\phi_1^2) - (\phi_1\phi_{2z}\sin^2\chi + \phi_2\phi_{1z}\cos^2\chi)]^2 + \phi_1^2\sin^2\chi + \phi_2^2\cos^2\chi\}^{-\frac{3}{2}}$$

$$\times \{(\phi_1^2\sin^2\chi + \phi_2^2\cos^2\chi)(\phi_{1zz}\phi_2\cos^2\chi + \phi_{2zz}\phi_1\sin^2\chi - [\theta_z(\phi_2^2-\phi_1^2)]_z\sin\chi\cos\chi$$

$$- \phi_1\phi_2\theta_z^2) - 2[(\phi_1\phi_{2z} - \phi_2\phi_{1z})\sin\chi\cos\chi - \theta_z(\phi_2^2\cos\chi + \phi_1^2\sin^2\chi)]$$

$$\times [(\phi_2\phi_{2z} - \phi_1\phi_{1z})\sin\chi\cos\chi + \phi_1\phi_2\theta_z]$$

$$- \phi_1\phi_2[(\phi_{1z}\cos\chi - \phi_2\theta_z\sin\chi)^2 + (\phi_{2z}\sin\chi + \phi_1\theta_z\cos\chi)^2 + 1]\} \quad , \tag{16}$$

and for convenience we have let ϕ_α stand for the triple $(\phi_\alpha, \phi_{\alpha z}, \phi_{\alpha zz})$ in the arguments of h and m. Also, we note the fact that m satisfies the condition

$$m(\phi_1, \phi_2, 0) = 0 \quad . \tag{17}$$

To complete the above system of equations, we must add an expression for $\dot{\theta}$, or the <u>rate of sectional rotation</u> of the jet. When the jet is non-circular,† this relation is

$$\dot{\theta} = \frac{\phi_2^2\omega_2 - \phi_1^2\omega_1}{\phi_2^2 - \phi_1^2} \quad . \tag{18}$$

For later reference, we record here the expression for the mechanical power P per unit mass of an incompressible jet, namely

$$\lambda P = \hat{\underset{\sim}{n}} \cdot \frac{\partial \underset{\sim}{v}}{\partial \xi} + \hat{\underset{\sim}{\pi}}^\alpha \cdot \underset{\sim}{w}_\alpha + \hat{\underset{\sim}{p}}^\alpha \cdot \frac{\partial \underset{\sim}{w}_\alpha}{\partial \xi} \quad , \tag{19}$$

where

$$\lambda = z'\rho^*\pi\phi_1\phi_2 \quad . \tag{20}$$

† At a point where the jet is circular ($\phi_1 = \phi_2$), there is no preferred orientation of the cross-section.

3. Small motions superposed on a uniform flow of an inviscid jet

First we consider an inviscid jet characterized by the constitutive assumptions

$$\hat{\underset{\sim}{n}} = \underset{\sim}{0} \ , \quad \hat{\underset{\sim}{\pi}}^{\alpha} = \underset{\sim}{0} \ , \quad \hat{\underset{\sim}{p}}^{\alpha} = \underset{\sim}{0} \ . \tag{21}$$

In this case, (10) to (12) reduce to the five scalar equations

$$\begin{aligned}
\pi\rho^{*}\phi_1\phi_2(v_t + vv_z) &= -p_z - \phi_2\phi_{1z}h(\phi_1,\phi_2,\theta_z) - \phi_1\phi_{2z}h(\phi_2,\phi_1,\theta_z) - \theta_z(\phi_2^2-\phi_1^2)m(\phi_1,\phi_2,\theta_z) \ , \\
\tfrac{1}{4}\pi\rho^{*}\phi_1^3\phi_2(\zeta_{1t} + v\zeta_{1z} + \zeta_1^2 - \omega_1^2) &= p + \phi_1\phi_2 h(\phi_1,\phi_2,\theta_z) \ , \\
\tfrac{1}{4}\pi\rho^{*}\phi_1\phi_2(\omega_{1t} + v\omega_{1z} + 2\zeta_1\omega_1) &= -m(\phi_1,\phi_2,\theta_z) \ , \\
\tfrac{1}{4}\pi\rho^{*}\phi_2^3\phi_1(\zeta_{2t} + v\zeta_{2z} + \zeta_2^2 - \omega_2^2) &= p + \phi_1\phi_2 h(\phi_2,\phi_1,\theta_z) \ , \\
\tfrac{1}{4}\pi\rho^{*}\phi_1\phi_2(\omega_{2t} + v\omega_{2z} + 2\zeta_2\omega_2) &= -m(\phi_2,\phi_1,\theta_z) \ ,
\end{aligned} \tag{22}$$

where we have expressed all functions in terms of the current position z of the material particle ξ at time t. The set (22) is completed by adding the incompressibility condition (9) together with the kinematic relations $(8)_{1,2}$ and (18). Consider now the simple solution

$$v = v_o \ , \quad \phi_1 = \phi_2 = a \ , \quad \zeta_1 = \zeta_2 = 0 \ , \quad \omega_1 = \omega_2 = 0 \ , \quad p = \pi a T \ , \tag{23}$$

which satisfies exactly the above system of differential equations and where v_o and a are constants. This solution represents a uniform circular jet moving with constant velocity v_o. With a suitable choice of reference frame we may let $v_o = 0$; then, due to the rotational symmetry of the directors in the plane of the cross-section, we may (at least for this solution) take

$$\theta_z = 0 \ , \tag{24}$$

without loss in generality.

We now examine small motions superposed on the uniform flow represented by the solution (23). This naturally leads to a discussion of jet instability and breakup (or disintegration) as each is generally understood in the literature. Accordingly, we shall determine a linearized version of the governing equations (22), (8), (9) and (18) appropriate for small deviations from the motion (23). We proceed in this manner owing to its relative simplicity and wide use, but note that any results arising from such a treatment shall be necessarily restricted by the limited scope of a linearized stability analysis.

Consider small deviations from the motion (23) in the form

$$\phi_\alpha = a + \tilde{\phi}_\alpha \ , \quad v = v_o + \tilde{v} \ , \quad \omega_\alpha = \tilde{\omega}_\alpha \ , \qquad p = \pi a T + \tilde{p} \tag{25}$$

and retain only linear terms in quantities represented by symbols with superposed tildas ("~") in all equations. Drawing upon the discussion preceding (24) we take $v_o = 0$ and $\theta_z = 0$ in the unperturbed flow. Then, in keeping with the linearized procedure, we assume that θ_z is small in the perturbed flow and neglect its square and product with quantities having tildas. If we linearize (16) in this manner, with the help of (14) and (15) we obtain

$$h = -\frac{\pi T}{a} + \frac{\pi T}{a^2}\left[\frac{3}{4} a^2\tilde{\phi}_{1zz} + \frac{1}{4} a^2\tilde{\phi}_{2zz} + \frac{1}{4}(\tilde{\phi}_2 - \tilde{\phi}_1) + \tilde{\phi}_2\right] \ , \quad m = 0 \ . \tag{26}$$

Introducing (25) and (26) into (22), (8) and (9), neglecting squares and products of small quantities and then dropping the tildas for simplicity, we are left with†

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†Since neither θ nor its derivatives appear in any of the other linearized equations (27), we do not record a linearized counterpart to (18).

$$\pi\rho^* a^2 v_t = -p_z + \pi T(\phi_1 + \phi_2)_z \quad ,$$

$$\tfrac{1}{4}\pi\rho^* a^4 \zeta_{1t} = p + \pi T[\tfrac{a^2}{4}(3\phi_1 + \phi_2)_{zz} - \phi_1 + \tfrac{1}{4}(\phi_2 - \phi_1)] \quad ,$$

$$\tfrac{1}{4}\pi\rho^* a^4 \zeta_{2t} = p + \pi T[\tfrac{a^2}{4}(3\phi_2 + \phi_1)_{zz} - \phi_2 + \tfrac{1}{4}(\phi_1 - \phi_2)] \quad , \tag{27}$$

$$\zeta_1 + \zeta_2 + v_z = 0 \quad ,$$

$$a\zeta_1 = \phi_{1t} \quad , \quad a\zeta_2 = \phi_{2t} \quad ,$$

$$\omega_{1t} = 0 \quad , \quad \omega_{2t} = 0 \quad .$$

In order to examine solutions to the system of partial differential equations (27), it is convenient to introduce the change of variables

$$\phi = \tfrac{1}{2}(\phi_1 + \phi_2) \quad , \quad \delta = \tfrac{1}{2}(\phi_1 - \phi_2) \quad . \tag{28}$$

Substituting (28) into (27) and then adding $(27)_2$ and $(27)_3$, with the help of $(27)_{5,6,7,8}$ we obtain

$$\rho^*\pi a^2 v_t = -p_z + 2\pi T\phi_z \quad ,$$

$$\tfrac{1}{4}\rho^*\pi a^3 \phi_{tt} = p_t + \pi T[a^2\phi_{zz} - \phi] \quad , \tag{29}$$

$$2\phi_t + av_z = 0 \quad .$$

Alternatively, if we use (28) in (27) and subtract $(27)_3$ from $(27)_2$, we have

$$\delta_{tt} = \frac{2T}{\rho^* a^3}(a^2\delta_{zz} - 3\delta) \quad . \tag{30}$$

Hence, in terms of the variables (28), the set (27) decouples into (29) and (30) and we can find solutions to each separately. First, we consider (29). Elimination of p and v among these equations yields

$$\tfrac{1}{4}a^2\phi_{ttzz} - 2\phi_{tt} = \frac{T}{\rho^* a}(\phi_{zz} + a^2\phi_{zzzz}) \tag{31}$$

as a differential equation in ϕ only. We examine solutions of (31) in the form

$$\phi(z,t) = f_o\ (k_o)\exp[i(\sigma_o t - k_o z)] \quad , \tag{32}$$

from which follows the dispersion relation

$$\sigma_o^2 = \frac{4T}{\rho^* a^3}\left(\frac{k_o^2 a^2 - 1}{8 + k_o^2 a^2}\right) k_o^2 a^2 \quad . \tag{33}$$

From (33) it is clear that the wave motion (32) is unstable for wave numbers satisfying

$$k_o^2 a^2 < 1 \quad . \tag{34}$$

Returning to (30), we consider periodic solutions to this equation in the form

$$\delta(z,t) = f_2(k_2)\exp[i(\sigma_2 t - k_2 z)] \quad . \tag{35}$$

The resulting dispersion relation is

$$\sigma_2^2 = \frac{2T}{\rho^* a^3}(k_2^2 a^2 + 3) \quad . \tag{36}$$

Now, from (28) we have for ϕ_1 and ϕ_2

$$\phi_1 = \phi + \delta \quad , \quad \phi_2 = \phi - \delta \quad , \tag{37}$$

so that the general motions consists of two parts: When $\delta = 0$, (37) gives $\phi_1 = \phi_2 = \phi$ and the circular cross-section of the jet remains circular in the perturbation. We call ϕ the symmetric mode. When $\phi = 0$, we have $\phi_1 = -\phi_2 = \delta$ and the cross-section of the perturbed jet is an oscillating ellipse, exchanging alternatively its major and minor axes. This we call the anti-symmetric mode.

In summary, a small disturbance to the motion (23) can be decomposed

into two modes of vibration: (i) a symmetric mode whose frequency is governed by (33) and (ii) an anti-symmetric mode whose frequency satisfies (36). The latter mode is stable for all wavelengths, while the symmetric mode is unstable for wavelengths satisfying (34). It is worth noting here that if the rotations ω_α and the twist θ_z are set equal to zero at the outset in the equations (10) to (13), then the results and conclusions of this section would remain unchanged.

Before closing this section, we make a comparison with the results of Rayleigh [1,2] who examined small deviations from the uniform flow of an inviscid, incompressible fluid in a straight circular jet. For this purpose, we introduce cylindrical polar coordinates (r,χ,z) such that the z-axis lies along the axis of the jet. Rayleigh considered disturbances from a circular jet of radius a in which the free surface had the modal forms

$$r = a + b(z,t)\cos n\chi \quad , \tag{38}$$

where n is an integer and b is small. With (38) as his basic assumption, Rayleigh examined solutions to the linearized three-dimensional equations for which b(z,t) has the form (32). The resulting dispersion relations for each integer n (as recorded in Lamb [14, §274]) are

$$\sigma_n^2 = \frac{T}{\rho^* a^3} \frac{I_n'(k_n a)[k_n^2 a^2 + n^2 - 1]k_n a}{I_n(k_n a)} \quad , \tag{39}$$

where I_n is the modified Bessel function of order n. The only value of n that can lead to unstable wave motion is $n = 0$, corresponding to an axially symmetric disturbance. With $n = 0$, (39) becomes

$$\sigma_o^2 = \frac{T}{\rho^* a^3} \frac{I_o'(k_o a)[k_o^2 a^2 - 1]k_o a}{I_o(k_o a)} \quad , \tag{40}$$

indicating a range of unstable wavelengths consistent with (34). The unbounded growth of disturbances in this range leads to an eventual disintegration of the

jet. For purposes of comparison, we tabulate in Table 1 values for σ_0^2 from (33) and (40) over the entire range of unstable wavelengths. Of the wavelengths in this range, the one that corresponds to the greatest magnitude of σ_0, and hence the most rapid rate of growth in the disturbance, will tend to dominate the disintegration process. The value obtained by Rayleigh for this wavelength corresponds to

$$k_0^2a^2 = 0.4858 \quad , \tag{41}$$

whereas from (33) we find

$$k_0^2a^2 = 0.4853 \tag{42}$$

yields maximum growth.

Setting $n=2$ in (39), we obtain the counterpart to (36) in the linearized three-dimensional theory, namely

$$\sigma_2^2 = \frac{T}{\rho^* a^3} \frac{I_2'(k_2a)[k^2a^2+3]k_2a}{I_2(k_2a)} \quad . \tag{43}$$

In line with (36), the motion governed by (43) is stable for all wavelengths. Expanding (43) in powers of $k_2^2a^2$ yields

$$\sigma_2^2 = \frac{2T}{\rho^* a^3} (k_2^2a^2+3)[1+\frac{1}{10}k_2^2a^2+\cdots] \quad . \tag{44}$$

A comparison of this relation with (36) suggests the appropriateness of the latter for fairly long wavelengths.

$k_o^2 a^2$	$i\sigma_o(\frac{T}{\rho_* a^3})^{-\frac{1}{2}}$	
	Directed Jet	Rayleigh
0.00	0.0000	0.0000
0.05	0.1536	0.1536
0.10	0.2107	0.2108
0.20	0.2793	0.2794
0.30	0.3181	0.3182
0.40	0.3381	0.3382
0.50	0.3429	0.3432
0.60	0.3341	0.3344
0.70	0.3107	0.3111
0.80	0.2696	0.2701
0.90	0.2010	0.2015
1.00	0.0000	0.0000

Table 1: Comparison of the frequency σ_o (in non-dimensional form) over the range (34) of unstable wavelengths as predicted by Eq. (33) of the direct theory of jets and by Eq. (40) due to Rayleigh [1,2].

4. A viscous elliptical jet without rotation or twist.

In this section we consider a jet of an incompressible linear viscous fluid; and, in view of the results for the stability of an inviscid jet, we limit the discussion to motions in which

$$\omega_1 = \omega_2 = 0 \quad , \quad \theta = \text{const.} \tag{45}$$

To account for the viscosity of the fluid medium, we must provide appropriate constitutive equations for the quantities $\hat{\mathbf{n}}$, $\hat{\boldsymbol{\pi}}^{\alpha}$ and $\hat{\mathbf{p}}^{\alpha}$. Much of the development of this section is similar to that of Green [10] in which the jet is restricted to be circular.

Referred to the orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ introduced in section 2, the response functions can be expressed in terms of their components in the form

$$\hat{\mathbf{n}} = \hat{n}^{i}\mathbf{e}_i \quad , \quad \hat{\boldsymbol{\pi}}^{\alpha} = \hat{\pi}^{\alpha i}\mathbf{e}_i \quad , \quad \hat{\mathbf{p}}^{\alpha} = \hat{p}^{\alpha i}\mathbf{e}_i \quad . \tag{46}$$

As a consequence of the symmetry of the assumed flow, (10) to (12) suggest that we put

$$\begin{gathered} \hat{n}^{1} = \hat{n}^{2} = 0 \quad , \\ \hat{\pi}^{12} = \hat{\pi}^{13} = \hat{\pi}^{21} = \hat{\pi}^{23} = 0 \quad , \\ \hat{p}^{12} = \hat{p}^{13} = \hat{p}^{21} = \hat{p}^{23} = 0 \quad . \end{gathered} \tag{47}$$

With the help of (4) to (6) and (46), the mechanical power (19) reduces to

$$\frac{\lambda}{z'}\,P = \hat{n}^{3}v_z + \hat{\pi}^{*11}\zeta_1 + \hat{\pi}^{*22}\zeta_2 + \hat{p}^{*11}\zeta_{1z} + \hat{p}^{*22}\zeta_{2z} \quad , \tag{48}$$

where we have set

$$\hat{\pi}^{*\alpha\alpha} = \phi_{\alpha}(\hat{\pi}^{\alpha\alpha}/z') + \phi_{\alpha z}\hat{p}^{\alpha\alpha} \quad , \quad \hat{p}^{*\alpha\alpha} = \phi_{\alpha}\hat{p}^{\alpha\alpha} \quad (\text{no sum on } \alpha) \quad . \tag{49}$$

We seek to characterize the linear viscous property of the fluid in appropriate constitutive equations for the one-dimensional functions

$$\hat{n}^3, \hat{\pi}^{*11}, \hat{\pi}^{*22}, \hat{p}^{*11}, \hat{p}^{*22} . \tag{50}$$

To this end, we assume that the quantities (50) are linear functions of degree one in the kinematic variables

$$v_z, \zeta_1, \zeta_2, \zeta_{1z}, \zeta_{2z} \tag{51}$$

with coefficients that depend upon ϕ_1 and ϕ_2. Hence, we take

$$\begin{aligned} \hat{n}^3 &= \lambda_1 v_z + \lambda_2^{\alpha}\zeta_{\alpha} + \lambda_3^{\alpha}\zeta_{\alpha z} , \\ \left.\begin{aligned} \hat{\pi}^{*\alpha\alpha} &= \lambda_4^{\alpha\beta}\zeta_{\beta} + \lambda_5^{\alpha\beta}\zeta_{\beta z} , \\ \hat{p}^{*\alpha\alpha} &= \lambda_6^{\alpha\beta}\zeta_{\beta} + \lambda_7^{\alpha\beta}\zeta_{\beta z} , \end{aligned}\right\} &\text{ (no sum on } \alpha) \end{aligned} \tag{52}$$

where $\lambda_1, \lambda_2^{\alpha}, \ldots, \lambda_7^{\alpha\beta}$ are functions of ϕ_1 and ϕ_2 and we have used (9) to eliminate v_z from the expressions for $\hat{\pi}^{*\alpha\alpha}$ and $\hat{p}^{*\alpha\alpha}$.

A three-dimensional linear viscous fluid is isotropic. In order that the one-dimensional theory under consideration reflect the symmetry properties of the fluid and the geometry of the jet, we impose the requirement that under the transformations

$$z \to -z \quad , \quad v \to -v \quad , \quad \zeta_{\alpha} \to \zeta_{\alpha} \quad , \quad \phi_{\alpha} \to \phi_{\alpha} \quad , \tag{53}$$

the mechanical power (48) remain invariant. Consequently, under (53) the functions (50) must transform according to

$$\hat{n}^3 \to \hat{n}^3 \quad , \quad \hat{\pi}^{*\alpha\alpha} \to \hat{\pi}^{*\alpha\alpha} \quad , \quad \hat{p}^{*\alpha\alpha} \to -\hat{p}^{*\alpha\alpha} . \tag{54}$$

Hence the relations (52) reduce to

$$\hat{n}^3 = \lambda_1 v_z + \lambda_2^\alpha \zeta_\alpha \ ,$$

$$\hat{\pi}^{*\alpha\alpha} = \lambda_4^{\alpha\beta}\zeta_\beta \quad (\text{no sum on } \alpha) \ , \tag{55}$$

$$\hat{p}^{*\alpha\alpha} = \lambda_7^{\alpha\beta}\zeta_{\beta z} \quad (\text{no sum on } \alpha) \ .$$

In order to determine explicit values for the coefficients in (55), we recall briefly some aspects of an approximation procedure for rod-like bodies in the three-dimensional theory. A detailed development of this procedure can be found in Green _et al_. [12] and a brief outline is included as an appendix to Caulk and Naghdi [11]. Without going into detail, we recall that the developments in Green _et al_. [12] are based on an approximation for the position vector to the material points in the rod-like body and involve integration* of the three-dimensional equations through its cross-section. Let the material points be identified with the convected coordinates θ^i $(i = 1,2,3)$ and, for convenience, set $\theta^3 = \xi$. Further, let $\mathbf{p}$ denote the position vector of a typical point at time t. Then,

$$\mathbf{p} = \mathbf{p}(\theta^\alpha, \xi, t) \ , \quad \mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \theta^i} \ ,$$
$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j \ , \quad \mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j \ , \quad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j \ , \quad g = \det g_{ij} \ , \tag{56}$$

where $\mathbf{g}_i$ and $\mathbf{g}^j$ are the covariant and contravariant base vectors, g_{ij} is the metric tensor, g^{ij} its inverse and δ^i_j is the Kronecker delta. In the present context the fluid is assumed to occupy a region of space in the neighborhood of the curve $\theta^\alpha = 0$, bounded by the free (material) surface

$$(\theta^1)^2 + (\theta^2)^2 = 1 \ , \tag{57}$$

where we identify $\theta^\alpha = 0$ with the z-axis. With the help of (4),

---

*It should be mentioned that the one-dimensional equations that result from this procedure can be brought into 1-1 correspondence with the theory of a directed curve.

the approximation for the position vector mentioned above leads to

$$\underset{\sim}{p} = z(\xi,t)\underset{\sim}{e}_3 + \theta^1\phi_1(\xi,t)\underset{\sim}{e}_1 + \theta^2\phi_2(\xi,t)\underset{\sim}{e}_2 \quad . \tag{58}$$

It follows from an examination of the second and third terms of (58) that (57) represents an elliptical cross-section with semiaxes ϕ_α. The velocity $\underset{\sim}{v}^*$, which is the material time derivative of $\underset{\sim}{p}$, is given by

$$\underset{\sim}{v}^* = v\underset{\sim}{e}_3 + \theta^1\phi_1\zeta_1\underset{\sim}{e}_1 + \theta^2\phi_2\zeta_2\underset{\sim}{e}_2 \quad , \tag{59}$$

where we have used (4), (8) and (45). For an incompressible linear viscous fluid, the determinate part of the stress response is given by

$$\underset{\sim}{T}^i = g^{\frac{1}{2}}\tau^{ij}\underset{\sim}{g}_j \quad , \quad \tau_{ij} = \mu[\underset{\sim}{v}^*_{,i} \cdot \underset{\sim}{g}_j + \underset{\sim}{v}^*_{,j} \cdot \underset{\sim}{g}_i] \quad , \tag{60}$$

where τ_{ij} and τ^{ij} are the covariant and contravariant components of the stress tensor, μ is the shear viscosity and a comma denotes partial differentiation with respect to θ^i. We may now use (57), (59) and (60) in the usual expressions† for the quantities (46) in terms of integrals over a cross-section of the rod-like body. The results of this rather long but routine calculation are

$$\hat{n}^3 = 2\mu\pi\,\phi_1\phi_2 v_z \quad ,$$

$$\left.\begin{aligned} \hat{\pi}^{*\alpha\alpha} &= 2\mu\pi\,\phi_1\phi_2\zeta_\alpha \quad , \\ \hat{p}^{*\alpha\alpha} &= \tfrac{1}{4}\,\mu\pi\,\phi_1\phi_2\phi_\alpha^2\zeta_{\alpha z} \quad , \end{aligned}\right\}\ (\textit{no sum on } \alpha) \tag{61}$$

so that we may identify

†See, for example, equation (A13) of Caulk and Naghdi [11].

$$\lambda_1 = 2\mu\pi\,\phi_1\phi_2 \ , \quad \lambda_2^{\alpha} = 0 \ , \quad \lambda_4^{11} = \lambda_4^{22} = 2\mu\pi\,\phi_1\phi_2 \ ,$$

$$\lambda_7^{11} = \tfrac{1}{4}\,\mu\pi\,\phi_2\phi_1^3 \ , \quad \lambda_7^{22} = \tfrac{1}{4}\,\mu\pi\,\phi_1\phi_2^3 \ , \quad \lambda_4^{12} = \lambda_4^{21} = \lambda_7^{12} = \lambda_7^{21} = 0 \tag{62}$$

in (55).

Adopting the values (62) and using the constitutive equations (55) in the field equations (10) to (13), we obtain the governing differential equations for a linear viscous jet, namely

$$\pi\rho^*\phi_1\phi_2(v_t + vv_z) = -p_z - \phi_2\phi_{1z}h(\phi_1,\phi_2) - \phi_1\phi_{2z}h(\phi_2,\phi_1) + 2\mu\pi(\phi_1\phi_2 v_z)_z \ ,$$

$$\tfrac{1}{4}\,\pi\rho^*\phi_1^3\phi_2(\zeta_{1t} + v\zeta_{1z} + \zeta_1^2) + 2\mu\pi\,\phi_1\phi_2\zeta_1 = p + \phi_1\phi_2 h(\phi_1,\phi_2) + \tfrac{1}{4}\,\mu\pi(\phi_1^3\phi_2\zeta_{1z})_z \ , \tag{63}$$

$$\tfrac{1}{4}\,\pi\rho^*\phi_2^3\phi_1(\zeta_{2t} + v\zeta_{2z} + \zeta_2^2) + 2\mu\pi\,\phi_1\phi_2\zeta_2 = p + \phi_1\phi_2 h(\phi_2,\phi_1) + \tfrac{1}{4}\,\mu\pi(\phi_2^3\phi_1\zeta_{2z})_z \ ,$$

where use has been made of (45) and (17) and we have let

$$h(\phi_1,\phi_2) = h(\phi_1,\phi_2,0) \ . \tag{64}$$

The set (63) is completed by adding (8) and (9). Apart from differences in notation, we note that $(63)_{1,2,3}$ reduce to those given by Green [10, Eqs. (6.3) and (6.4)] in the special case of a circular cross-section $(\phi_1 = \phi_2)$ and in the absence of gravity.

5. Small motions superposed on a uniform flow of a viscous jet.

The motion (23) satisfies the differential equations (63), (8) and (9); and, hence, it also represents an exact solution for the viscous jet discussed in section 4. Since in the case of an inviscid jet (see section 3) the superposed rotations $\tilde{\omega}_\alpha$ had no effect on the resulting differentiation equations, for simplicity we assume that $\tilde{\omega}_\alpha = 0$ here and consider small motions superposed on the uniform flow (23) in the form specified by the first two of (25) and the fourth of (25). In a manner similar to that employed in section 3, we again neglect squares and products of quantities represented by symbols with superposed tildas in (63), (8) and (9). After setting $v_o = 0$ without loss in generality, the resulting linearized equations are

$$
\begin{gathered}
\pi\rho^* a^2 v_t = -p_z + \pi T(\phi_1 + \phi_2)_z + 2\mu\pi a^2 v_{zz} \quad , \\
\tfrac{1}{4}\pi\rho^* a^4 \zeta_{1t} + 2\mu\pi a^2 \zeta_1 = p + \pi T[\tfrac{a^2}{4}(3\phi_1 + \phi_2)_{zz} - \phi_1 + \tfrac{1}{4}(\phi_2 - \phi_1)] + \tfrac{1}{4}\mu\pi a^4 \zeta_{1zz} \quad , \\
\tfrac{1}{4}\pi\rho^* a^4 \zeta_{2t} + 2\mu\pi a^2 \zeta_2 = p + \pi T[\tfrac{a^2}{4}(3\phi_2 + \phi_1)_{zz} - \phi_2 + \tfrac{1}{4}(\phi_1 - \phi_2)] + \tfrac{1}{4}\mu\pi a^4 \zeta_{2zz} \quad , \\
v_z + \zeta_1 + \zeta_2 = 0 \quad , \quad \phi_{1t} = a\zeta_1 \quad , \quad \phi_{2t} = a\zeta_2 \quad ,
\end{gathered} \tag{65}
$$

where again for convenience the tildas have been omitted.

Again we utilize the change of variables (28) and by adding $(65)_2$ and $(65)_3$, with the help of $(65)_{4,5,6}$, we obtain

$$
\begin{gathered}
\pi\rho^* a^2 v_t = -p_z + 2\pi T\phi_z + 2\mu\pi a^2 v_{zz} \quad , \\
\tfrac{1}{4}\pi\rho^* a^3 \phi_{tt} + 2\mu\pi a \phi_t = p + \pi T[a^2\phi_{zz} - \phi] + \tfrac{1}{4}\mu\pi a^3 \phi_{tzz} \quad , \\
2\phi_t + a v_z = 0 \quad .
\end{gathered} \tag{66}
$$

Subtraction of $(65)_3$ from $(65)_2$, after using (28) and $(65)_{5,6}$, yields

$$\frac{1}{2}\rho^* a^3 \delta_{tt} + 4\mu a \delta_t = T(a^2\delta_{zz} - 3\delta) + \frac{1}{2}\mu a^3 \delta_{tzz} \quad . \tag{67}$$

Thus, the linearized system (65) is decoupled through the change of variables (28), just as in the case of the inviscid jet. Again the solution of (65) will have the form (37) and decompose into a symmetric mode ϕ and an anti-symmetric mode δ. Eliminating p and v among $(66)_{1,2,3}$, for the symmetric mode we obtain

$$\frac{1}{4} a^2\phi_{ttzz} - 2\phi_{tt} = \frac{\mu}{\rho^*}\left(\frac{1}{4} a^2\phi_{zzzz} - 6\phi_{zz}\right)_t + \frac{T}{\rho^* a}\left(a^2\phi_{zzzz} + \phi_{zz}\right) \quad . \tag{68}$$

For solutions of (68) to be of the form (32), σ_o and k_o must satisfy

$$(i\sigma_o)^2 + \frac{\frac{\mu}{a^2\rho^*}\left(\frac{1}{8}k_o^2a^2 + 3\right)k_o^2a^2}{\frac{1}{8}k_o^2a^2 + 1}(i\sigma_o) = \frac{\frac{T}{2\rho^* a^3}\left(1 - k_o^2a^2\right)k_o^2a^2}{\frac{1}{8}k_o^2a^2 + 1} \quad . \tag{69}$$

It follows from (69) that $i\sigma_o$ can be real and positive if and only if

$$k_o^2a^2 < 1 \quad . \tag{70}$$

Hence the range of unstable wavelengths for the symmetric mode is precisely the same as for an inviscid jet. The effect of viscosity in the present case is to diminish the magnitude of σ_o over the range (70) and therefore retard the impending disintegration of the jet. As in section 3, we consider solutions for the anti-symmetric mode in the form (35). Using (35) in (67) we obtain the relation

$$(i\sigma_2)^2 + \frac{\mu}{\rho^* a^2}\left(k_2^2a^2 + 8\right)(i\sigma_2) + \frac{2T}{\rho^* a^3}\left(k_2^2a^2 + 3\right) = 0 \tag{71}$$

between σ_2 and k_2. From (71) one can show that $i\sigma_2$ has a negative real part for all values of k_2. This indicates, as in the case of the inviscid jet, that the anti-symmetric mode is stable for disturbances of all wavelengths. The effect of viscosity, however, is to damp the disturbance in proportion

to the magnitude of the shear viscosity μ. Critical damping for a given value of k_2 corresponds to

$$A^2 = \frac{2\mu^2}{\rho^* Ta} = \frac{16(k_2^2 a^2 + 3)}{(k_2^2 a^2 + 8)^2} \quad , \tag{72}$$

where we have introduced the non-dimensional parameter A for later convenience.*

We close with a comparison of certain results of this section with those of a similar investigation by Weber [4] who has examined small axially symmetric perturbations to uniform flow of a cylindrical viscous jet using an approximate form of the linearized Navier-Stokes equations. Weber's procedure employs specific assumptions on the variation of stress and velocity in the cross-section of the jet and ignores all but the axial component of momentum. This approach leads to a one-dimensional reduction of the three-dimensional equations and corresponding to (69) gives

$$(i\sigma_o)^2 + \frac{\mu}{\rho^* a^2} 3k_o^2 a^2 (i\sigma_o) = \frac{T}{2\rho^* a^3} (1 - k_o^2 a^2) k_o^2 a^2 \quad . \tag{73}$$

A plot of $i\sigma_o$ versus $k_o a$ is given in Fig. 1 for various values of the parameter A over the range (70) of unstable wavelengths, using both (69) and (73). It can be seen from this graph that the difference between the results of each approach is greatest for an inviscid jet $(A = 0)$ and gradually diminishes with increasing viscosity, other things being equal.

On the basis of the close agreement (Table 1) with the exact three-dimensional analysis of Rayleigh [1,2] for an inviscid jet $(A = 0)$, it is reasonable to infer from Fig. 1 that for a viscous jet the results of this section constitute an improvement over the approximate treatment of Weber [4]. In support of this inference, we appeal to some numerical results recorded in

*The parameter A^2 can be recognized as twice the ratio of the Weber number to the square of the Reynold's number.

Chandrasekhar [6] which are based on the implicit dispersion relation obtained from the linearized Navier-Stokes equations.[†] Figure 2 shows a magnified portion of one of the curve pairs in Fig. 1 corresponding to $A = 0.5$ along with points obtained from the tables in Chandrasekhar [6] . We show the region near maximum σ_0 in view of its importance in the breakup process. The fact that the theory of a directed fluid jet offers an improvement over Weber's results is clearly evident.

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[†] The difficulty in dealing analytically with this same dispersion relation is what led Weber [4] to consider an approximation to the three-dimensional equations.

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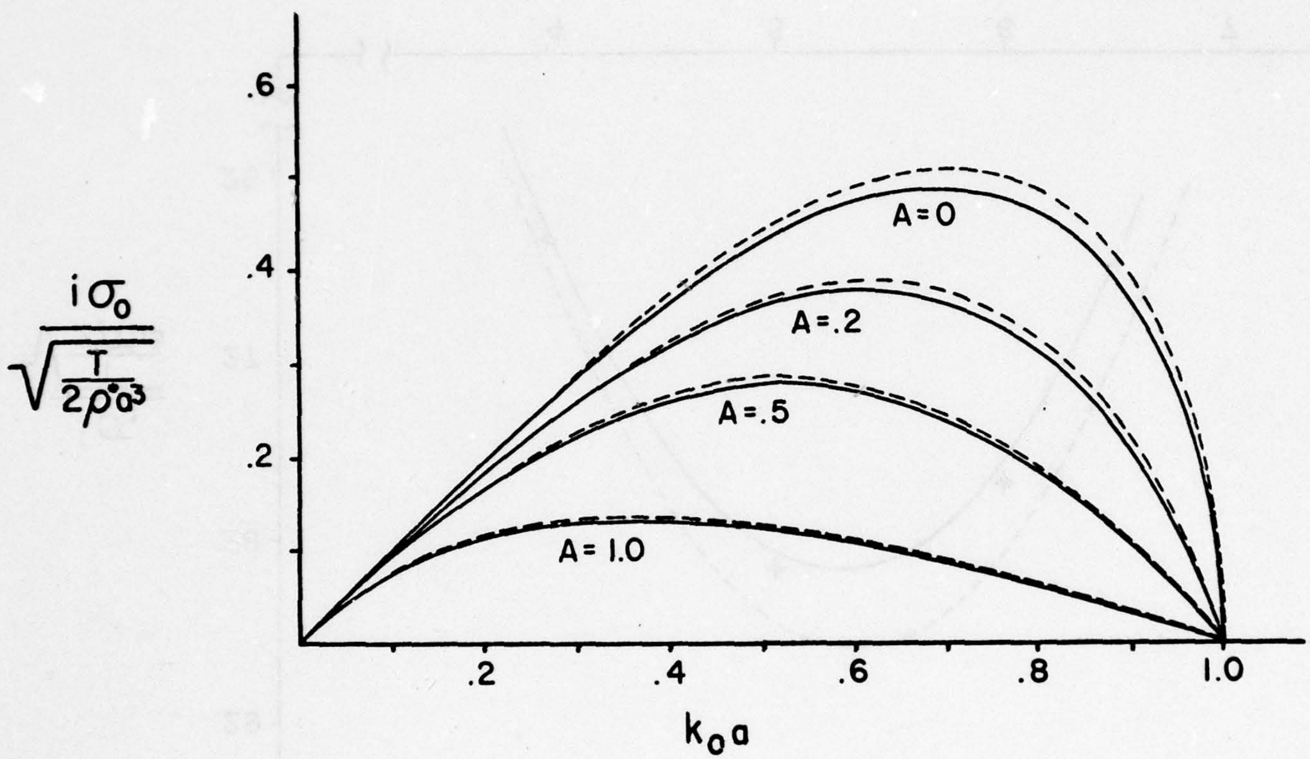

Fig. 1. Comparison of the dispersion relation obtained by the direct approach (——) with Weber's [4] approximation (----) over the range of unstable wavelengths for various values of the nondimensional parameter A defined by equation (72). The curves for A = 0 correspond to those of an inviscid jet. Comparison of the present solution for A = 0 with Rayleigh's [1,2] exact three-dimensional solution is not shown due to the close agreement indicated in Table 1.

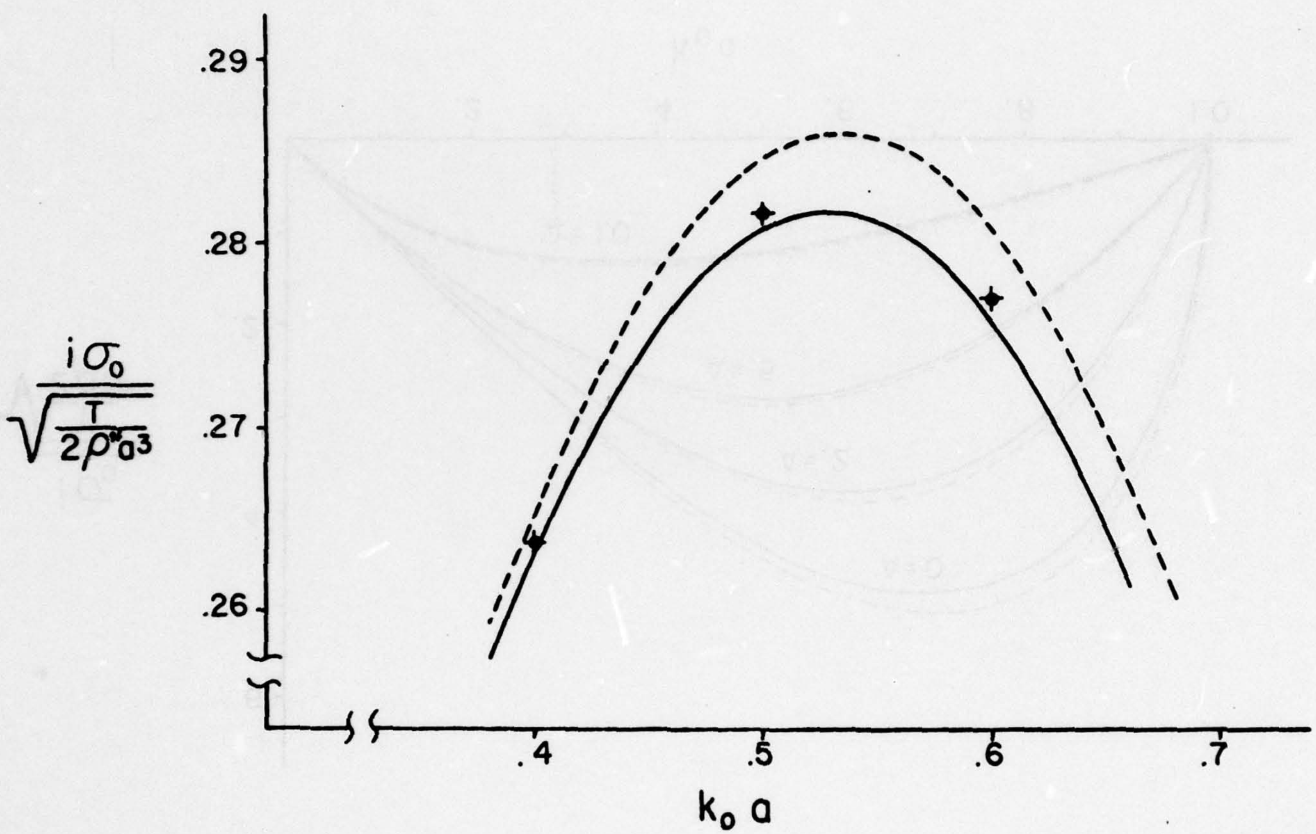

Fig. 2. Comparison of the dispersion relation obtained by the direct approach (——) with Weber's [4] approximation (----) for a magnified portion of one pair of the curves ($A = 0.5$) in Fig. 1. Also shown are three points obtained by interpolation from Chandraskhar's ([6], p. 544) Tables when $A = 0.5$.

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Instability of jets, incompressible, inviscid and viscous fluids, direct approach, surface tension, one-dimensional theory of a directed fluid jet, differential equations for small motion on uniform flow, comparison with available results from the three-dimensional theory.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper is concerned with the instability of inviscid and viscous jets utilizing the basic equations of the one-dimensional direct theory of a fluid jet based on the concept of a Cosserat (or a directed) curve. First, a system of differential equations is derived for small motions superposed on uniform flow of an inviscid straight circular jet which can twist along its axis. Periodic wave solutions are then obtained

(continued)

20. Abstract (continued)

for this system of linear equations; and, with reference to a description of growth in the unstable mode, the comparison of the resulting dispersion relation is found to agree extremely well with the classical (three-dimensional) results of Rayleigh. Next, constitutive equations are obtained for a viscous elliptical jet and these are used to discuss both the symmetric and the anti-symmetric small disturbances in the shape of the free surface of a circular jet. Through a comparison with available three-dimensional numerical results, the solution obtained is shown to be an improvement over an existing approximate solution of the problem.